Linear Algebra I

14/12/2020

1 Systems of linear equations $(4+1+10+5=20 \text{ pts})$

Two groups of students attend lectures in the same classroom. The first group consists of p students and the second group consists of q students. In the classroom, there are r rows of seats. When the first group follows a lecture, there are 6 students sitting at each row except the last row which is occupied only by 4 students. When the second group follows another lecture, there are 5 students sitting at each row except the last row which is occupied by 6 students. When both groups follow yet another lecture, there are 10 students sitting at each row and 10 more standing.

(a) Find a system of linear equations in the unknowns p, q, and r describing the above scenario.

(b) Write down the augmented matrix.

(c) By performing elementary row operations, put the augmented matrix into **reduced** row echelon form.

(d) Determine the solution set.

REQUIRED KNOWLEDGE: **Gauss elimination, row operations, reduced row echelon form, notions of lead/free variables.**

SOLUTION:

1a: We can obtain three linear equations from what we are given:

$$\begin{aligned} p &= 6(r-1)+4 \\ q &= 5(r-1)+6 \\ p+q &= 10r+10. \end{aligned}$$

By putting these equations in the standard form, we arrive at:

$$\begin{aligned} p - 6r &= -2 \\ q - 5r &= 1 \\ p+q-10r &= 10. \end{aligned}$$

1b: The augmented matrix is given by

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 1 & 1 & -10 & 10 \end{array}\right].$$

1c: By applying elementary row operations, we can proceed as follows:

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 1 & 1 & -10 & 10 \end{array}\right] \xrightarrow{(3) \leftarrow (3) - 1\cdot(1)} \left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -4 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 0 & 1 & -4 & 12 \end{array}\right] \xrightarrow{(3) \leftarrow (3) - 1\cdot(2)} \left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 11 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & -2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 11 \end{array}\right] \xrightarrow[(1) \leftarrow (1) + 6\cdot(3)]{(2) \leftarrow (2) + 5\cdot(3)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 64 \\ 0 & 1 & 0 & 56 \\ 0 & 0 & 1 & 11 \end{array}\right]$$

1d: All variables are lead variables. So, there is unique solution:

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 64 \\ 56 \\ 11 \end{bmatrix}.$$

In other words, there are 64 students in the first group, 56 in the second group and the classroom has 11 rows.

2 Matrix multiplication (20 pts)

Let λ be a scalar. Consider the matrix

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

By mathematical induction on k, show that

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{(k-1)k}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

for all $k \geqslant 2$.

Required Knowledge: **Matrix multiplication, mathematical induction.**

Solution:

To prove the statement by induction on k, we begin with the case $k = 2$. Note that

$$J^2 = JJ = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

Therefore, we see that the statement holds for $k = 2$. Assume, now, that the statement holds for some $k \geqslant 2$, i.e.

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{(k-1)k}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}.$$

Note that

$$J^{k+1} = JJ^k = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{(k-1)k}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k & (\frac{(k-1)k}{2} + k)\lambda^{k-1} \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{bmatrix}.$$

Since $\frac{(k-1)k}{2} + k = \frac{k(k+1)}{2}$, we have

$$J^{k+1} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k & \frac{k(k+1)}{2}\lambda^{k-1} \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{bmatrix}.$$

Hence, we see that the statement holds for $k + 1$. By induction, then, it holds for all $k \geqslant 2$.

3 Nonsingular matrices (5 + 12 + 8 = 25 pts)

Let a be a real number and $x \in \mathbb{R}^n$ be a vector with $x^T x = 1$. Consider

$$M(a) = I_n + axx^T.$$

(a) Let a, b be given. Show that there exists a real number c such that $M(a)M(b) = M(c)$.

(b) Show that $M(a)$ is nonsingular if and only if $a \neq -1$.

(c) Find the inverse of $M(a)$ for $a \neq -1$.

Required Knowledge: **Matrix multiplication, nonsingular matrices, and inverse matrix.**

Solution:

3a: Note that

$$M(a)M(b) = (I_n + axx^T)(I_n + bxx^T) = I_n + bxx^T + axx^T + abxx^T xx^T = I_n + (a+b+ab)xx^T$$

where the last equality is obtained by using $x^T x = 1$. Therefore, we see that $c = a + b + ab$ proves the claim.

3b: For the 'if' part, we can use the previous subproblem. Indeed, we see that $M(a)M(\frac{-a}{1+a}) = M(0) = I_n$. As such, $M(a)$ is nonsingular. For the 'only if' part, we can prove the contrapositive, that is $M(-1)$ is singular. Indeed, $M(-1)x = (I_n - xx^T)x = x - xx^T x = x - x = 0$. Since $x \neq 0$, $M(-1)$ is singular.

3c: In the previous subproblem, we made the observation that $M(a)M(\frac{-a}{1+a}) = M(0) = I_n$ whenever $a \neq -1$. Therefore, we have

$$M(a)^{-1} = M(\frac{-a}{1+a}) = I_n - \frac{a}{1+a}xx^T.$$

4 Determinants (20 + 5 = 25 pts)

Let a, b, c, d be scalars. Consider the matrix

$$M = \begin{bmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{bmatrix}.$$

(a) Find $\det M$.

(b) Suppose that $b = c = d = 1$. Determine all values of a such that M is **singular**.

REQUIRED KNOWLEDGE: **Determinants, cofactor expansion, elementary row operations.**

SOLUTION:

4a: By performing elementary row operations, we can obtain

$$\begin{bmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{bmatrix} \xrightarrow[\textcircled{4} \leftarrow \textcircled{4} - 1\cdot\textcircled{1}]{\textcircled{2} \leftarrow \textcircled{2} - 1\cdot\textcircled{1} \quad \textcircled{3} \leftarrow \textcircled{3} - 1\cdot\textcircled{1}} \begin{bmatrix} 1+a & 1 & 1 & 1 \\ -a & b & 0 & 0 \\ -a & 0 & c & 0 \\ -a & 0 & 0 & d \end{bmatrix}$$

Since we have applied on type III operations, we see that

$$\det M = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ -a & b & 0 & 0 \\ -a & 0 & c & 0 \\ -a & 0 & 0 & d \end{vmatrix}.$$

Cofactor expansion along the first row results in:

$$\begin{aligned} \begin{vmatrix} 1+a & 1 & 1 & 1 \\ -a & b & 0 & 0 \\ -a & 0 & c & 0 \\ -a & 0 & 0 & d \end{vmatrix} &= (1+a)\begin{vmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{vmatrix} - \begin{vmatrix} -a & 0 & 0 \\ -a & c & 0 \\ -a & 0 & d \end{vmatrix} + \begin{vmatrix} -a & b & 0 \\ -a & 0 & 0 \\ -a & 0 & d \end{vmatrix} - \begin{vmatrix} -a & b & 0 \\ -a & 0 & c \\ -a & 0 & 0 \end{vmatrix} \\ &= (1+a)bcd + acd + abd + abc \\ &= abcd + abc + abd + acd + bcd \end{aligned}$$

where to obtain the second line we can use the fact that the first two determinants follows from the triangular structure whereas the last two can be obtained by cofactor expansion along the second and the third row, respectively.

4b: If $b = c = d = 1$, then we have $\det M = 4a + 1$. Consequently, M is singular if and only if $a = \frac{-1}{4}$.